HW 4-5 WAVE EQUATIONS FILIPPO NARDI

SPIN 1/2

A massive spin 1/2 particle can be interpolated by a Dirac spinor

$$\psi(x;P,\lambda) = u(P,\lambda)e^{-ip\cdot x} = \langle 0|\psi(x)|P,\lambda\rangle$$

where $\lambda = \pm\frac{1}{2}$. We recall the expressions for the spinor wave functions in momentum space

$$u(P,\lambda) = \begin{pmatrix} \omega_{-2\lambda}\chi_{2\lambda}(\hat{P}) \\ \omega_{2\lambda}\chi_{2\lambda}(\hat{P}) \end{pmatrix}$$

where $\omega_\pm = \sqrt{E\pm|\vec{P}|}$ and

$$\chi_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \qquad \chi_- = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

In the rest frame $(\bar{P}^\mu = (m,\vec{0}))$

$$u(\bar{P},\lambda) = \sqrt{m}\begin{pmatrix} \chi_{2\lambda}(\hat{z}) \\ \chi_{2\lambda}(\hat{z}) \end{pmatrix}$$

where

$$\chi_+(\hat{z}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_-(\hat{z}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We now notice that, as a consequence of the fact that Dirac spinors are a basis for a representation of parity, and the state of a particle at rest, as the vacuum, is invariant under parity, we get

$$u(\bar{P},\lambda) = \langle 0|\psi(0)|\bar{P},\lambda\rangle = \langle 0|U^\dagger(P)\,\psi(0)\,U(P)\,U^\dagger(P)|\bar{P},\lambda\rangle =$$

$$= \gamma^0\langle 0|\psi(0)|\bar{P},\lambda\rangle = \gamma^0 u(\bar{P},\lambda)$$

Therefore we find trivially

$$(m\gamma^0 - m)\,u(\bar{P},\lambda) = 0$$

Alternatively we could expliaitely use the expressions for the spinors, in order to find the equality

$$\gamma^0 u(\bar{P},\lambda) = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{m}\,\chi_{2\lambda} \\ \sqrt{m}\,\chi_{2\lambda} \end{pmatrix} = \begin{pmatrix} \sqrt{m}\,\chi_{2\lambda} \\ \sqrt{m}\,\chi_{2\lambda} \end{pmatrix} = u(\bar{P},\lambda)$$

We now notice that we can rewrite the equation above using the four vector $\bar{P}^\mu = (m,\vec{0})$ as

$$(\bar{P}_\mu\gamma^\mu - m)\,u(\bar{P},\lambda) = 0$$

We recognize this equation as the Dirac equation in momentum space for the wave function in the rest frame. We can generalize the equation to an arbitrory frame using the appropiate Lorentz transformation. In fact

$$u(P,\lambda) = \langle 0|\Psi(0)|P,\lambda\rangle = \langle 0|\Psi(0)\,U(\Lambda_p)|\bar{P},\lambda\rangle =$$

$$= \langle 0|U^{+}(\Lambda_p)\,\Psi(0)\,U(\Lambda_p)|\bar{P},\lambda\rangle = S(\Lambda_p)\,\langle 0|\Psi(0)|\bar{P},\lambda\rangle =$$

$$= S(\Lambda_p)\,u(\bar{P},\lambda)$$

where $S(\Lambda_p)$ is the representation for the Lorentz transformation on Dirac Spinors which is given by

$$S(\Lambda_p) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) \qquad J^{\mu\nu} = \frac{i}{4}[\gamma^{\mu},\gamma^{\nu}]$$

Let us recall the property

$$S^{-1}(\Lambda_p)\,\gamma^{\mu}\,S(\Lambda_p) = \Lambda_{p}{}^{\mu}{}_{\nu}\,\gamma^{\nu}$$

Thus, we can multiply the Dirac equation in the rest frame with $S(\Lambda)$ on the left, to get

$$S(\Lambda)(\bar{P}_{\mu}\gamma^{\mu} - m)\,u(\bar{P},\lambda) = S(\Lambda_p)(\bar{P}_{\mu}\gamma^{\mu} - m)\,S^{-1}(\Lambda_p)\,S(\Lambda_p)\,u(\bar{P},\lambda) =$$

$$= (\bar{P}_{\mu}(\Lambda_p^{-1})^{\mu}{}_{\nu}\,\gamma^{\nu} - m)\,u(P,\lambda) =$$

$$= ((\Lambda_p\bar{P})_{\mu}\,\gamma^{\mu} - m)\,u(P,\lambda) =$$

$$= (P_{\mu}\gamma^{\mu} - m)\,u(P,\lambda)$$

Therefore, we find the Dirac equation in momentum space

$$(\not{P} - m)\,u(P,\lambda) = 0$$

At this point we would like to Fourier transform the result in position space. For this purpose, we notice that given a differential operator $D(\partial)$, we can define the operator in Fourier space which acts multiplicatively as

$$\Pi(P) = e^{+ip\cdot x}\,D(\partial)\,e^{-ip\cdot x}$$

Therefore, we find

$$\not{P} - m = e^{+ip\cdot x}\,(i\not{\partial} - m)\,e^{-ip\cdot x}$$

and we finally get the Dirac equation in position space

$$(i\not{\partial} - m)\,\psi(x;P,\lambda) = 0$$

SPIN 1

A massive spin 1 particle can be interpolated by a vector field

$$\psi^\mu(x;p,\lambda) = \mathcal{E}^\mu(p,\lambda)\, e^{-ip\cdot x} = \langle 0 | A^\mu(x) | p,\lambda \rangle$$

where $\lambda = -1, 0, +1$ and

$$\mathcal{E}^\mu(p,+) = -\mathcal{E}^\mu(p,-)^* = \frac{e^{i\phi}}{\sqrt{2}} \begin{pmatrix} 0 \\ -\cos\theta\cos\phi + i\sin\phi \\ -i\cos\phi - \cos\theta\sin\phi \\ \sin\theta \end{pmatrix} \qquad \mathcal{E}^\mu(p,0) = \begin{pmatrix} \frac{|\vec{p}|}{m} \\ \frac{E}{m}\hat{p} \end{pmatrix}$$

In the particle rest frame

$$\mathcal{E}^\mu(\bar{p},+) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} \qquad \mathcal{E}^\mu(\bar{p},-) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ +1 \\ -i \\ 0 \end{pmatrix} \qquad \mathcal{E}^\mu(p,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We notice that in all cases

$$\mathcal{E}^0(\bar{p},\lambda) = 0$$

This condition can be rewritten using the fact that $\bar{p}^\mu = (m, \vec{0})$ as

$$\bar{p}_\mu \mathcal{E}^\mu(\bar{p},\lambda) = 0$$

Proceeding in exactly the same way we have done in the previous point we can generalize the above constraint in a generic frame.
In fact

$$\begin{aligned} \mathcal{E}^\mu(p,\lambda) &= \langle 0 | A^\mu(0) | p,\lambda \rangle = \langle 0 | A^\mu(0)\, U(\Lambda_p) | \bar{p},\lambda \rangle = \\ &= \langle 0 | U^\dagger(\Lambda_p) A^\mu(0) U(\Lambda_p) | \bar{p},\lambda \rangle = \Lambda^\mu_{p\,\nu} \langle 0 | A^\nu(0) | \bar{p},\lambda \rangle = \\ &= \Lambda^\mu_{p\,\nu}\, \mathcal{E}^\nu(\bar{p},\lambda) \end{aligned}$$

Therefore

$$\bar{p}_\mu \mathcal{E}^\mu(\bar{p},\lambda) = (\Lambda_p \bar{p})_\mu (\Lambda_p \mathcal{E}(\bar{p},\lambda))^\mu = p_\mu \mathcal{E}^\mu(p,\lambda) = 0$$

In addition, we have the trivial constraint

$$(-p^2 + m^2)\, \mathcal{E}^\mu(p,\lambda) = 0$$

coming from the fact that $p^\mu = (\sqrt{m^2 + |\vec{p}|^2}, \vec{p})$.

Now, we can use again the relation $\Pi(p) = e^{+ip\cdot x} D(\partial)\, e^{-ip\cdot x}$

to rewrite the constraints in the form

$$(\Box + m^2)\psi^\mu(x;p,\lambda) = 0$$

$$\partial_\mu \psi^\mu(x;p,\lambda) = 0$$

1. We now consider the definitions

$$\Pi^L_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box}$$

$$\Pi^T_{\mu\nu} = \eta_{\mu\nu} - \Pi^L_{\mu\nu}$$

These operators are projectors, that is, they satisfy

$$\Pi^L \cdot \Pi^L = \Pi^L$$

$$\Pi^T \cdot \Pi^T = \Pi^T$$

$$\Pi^T \cdot \Pi^L = 0$$

$$\Pi^{T\mu}{}_\nu + \Pi^{L\mu}{}_\nu = \delta^\mu_\nu$$

We now show that the two constraints above are equivalent to just one equation

$$[(\Box + m^2)\Pi^T_{\mu\nu} + \alpha \Pi^L_{\mu\nu}]\psi^\mu(x;p,\lambda) = 0$$

We start proving that the validity of

$$(\Box + m^2)\psi^\mu(x;p,\lambda) = 0$$

$$\partial_\mu \psi^\mu(x;p,\lambda) = 0$$

implies the equation above. In fact,

$$[(\Box + m^2)\Pi^T_{\mu\nu} + \alpha \Pi^L_{\mu\nu}]\psi^\nu(x;p,\lambda) =$$

$$= [(\Box + m^2)\eta_{\mu\nu} - (\Box + m^2 - \alpha)\frac{\partial_\mu \partial_\nu}{\Box}]\psi^\nu(x;p,\lambda) =$$

$$= (\Box + m^2)\psi_\mu(x;p,\lambda) - (\Box + m^2 - \alpha)\frac{\partial_\mu}{\Box}\partial_\nu \psi^\nu(x;p,\lambda) = 0$$

On the contrary, if

$$[(\Box+m^2)\Pi^T_{\mu\nu}+\alpha\Pi^L_{\mu\nu}]\psi^\nu(x;P,\lambda)=0$$

then multiplying on the left with Π^T and Π^L, using the fact that they are projectors we find

$$(\Box+m^2)\,\Pi^{T\mu}{}_{\nu}\,\psi^\nu(x;P,\lambda)=0$$

$$\Pi^{L\mu}{}_{\nu}\,\psi^\nu(x;P,\lambda)=0$$

These relations are equivolent to the two constraints above. In fact contracting the secon equation with ∂_μ we find

$$\partial_\nu\psi^\nu(x;P,\lambda)=0$$

and the first one becomes

$$(\Box+m^2)\psi^\mu(x;P,\lambda)=0$$

Thus, the equivalence is proved.

2. We now want to find the value of α for wich the equation

$$[(\Box+m^2)\Pi^T_{\mu\nu}+\alpha\Pi^L_{\mu\nu}]\psi^\nu(x;P,\lambda)=0$$

is local. For this purpose it is enaugh to consider the very definition of the projectors

$$[(\Box+m^2)\eta_{\mu\nu}+(-\Box-m^2+\alpha)\frac{\partial_\mu\partial_\nu}{\Box}]\psi^\nu(x;P,\lambda)=0$$

We find that choosing $\alpha=m^2$ the equation becomes local

$$[(\Box+m^2)\delta^\mu{}_\nu-\partial^\mu\partial_\nu]\psi^\nu(x;P,\lambda)=0$$

This is the Proca equation.

SPIN 2

We now consider a massive spin 2 particle whose states are $|p,\lambda\rangle$ with $\lambda = -2,-1,0,1,2$. A symmetric rank two tensor can interpolate between these states and the vacuum. Thus, we define the wave function

$$\psi^{\mu\nu}(x) = \mathcal{E}^{\mu\nu}(p,\lambda)\, e^{-ip\cdot x} = \langle 0|\, h^{\mu\nu}(x)\, |p,\lambda\rangle$$

1. We want to find the polarizations $\mathcal{E}^{\mu\nu}(\bar{p},\lambda)$ in the rest frame $(\bar{p}^\mu = (m,\vec{0}))$. We notice that the rank two tensor $h^{\mu\nu}$ transforms in a reducible representation of the Lorentz group.

$$h^{\mu\nu} \sim (1,1) \oplus (0,0) \overset{SO(3)}{\sim} 0 \oplus 0 \oplus 1 \oplus 2$$

In fact, we can always write $h^{\mu\nu} = \bar{h}^{\mu\nu} + 1/4\, h\, \eta^{\mu\nu}$ where $h = h^\alpha{}_\alpha \sim (0,0)$ and $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{4} h\, \eta^{\mu\nu} \sim (1,1)$ In addition $(0,0) \overset{SO(3)}{\sim} 0$, $(1,1) \overset{SO(3)}{\sim} 0 \oplus 1 \oplus 2$.

Let us identify the irreducible components of the tensor. We recall the representation for rotation matrix

$$\Lambda(R) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right)$$

and

$$h'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma h^{\rho\sigma}$$

Therefore
(by convention, latin indices run over 1,2,3 while greek ones over 0,1,2,3)

$$h'^{00} = h^{00} \qquad \text{spin 0 component}$$

$$h'^i{}_i = h^i{}_i \qquad \text{spin 0 component}$$

$$h'^{0i} = R^i{}_j h^{0j} \qquad \text{spin 1 component}$$

$$h'^{ij} = R^i{}_k R^j{}_l h^{kl} \qquad \text{spin 2 component}$$

Thus, we have identified the spin 2 component of the rank two tensor. This means that, for a spin 2 particle, the polarization tensors satisfy the constraints

$$\mathcal{E}^{0\mu}(\bar{p},\lambda) = \langle 0|\, h^{0\mu}(0)\, |\bar{p},\lambda\rangle = \mathcal{E}^{\mu 0}(\bar{p},\lambda) = 0$$

$$\mathcal{E}^\mu{}_\mu(\bar{p},\lambda) = \langle 0|\, h^\mu{}_\mu(0)\, |\bar{p},\lambda\rangle = 0$$

and the non zero components are given by $\mathcal{E}^{ij}(\bar{P},\lambda)$.
Let us now derive the polarization tensors. For this purpose, let us consider the SO(3) generators: $J^i \otimes 1\!\!1 + 1\!\!1 \otimes J^i$. We can derive the representation acting with the lowering operator $J_- \otimes 1\!\!1 + 1\!\!1 \otimes J_-$ (where $J_\pm = J^1 \pm i J^2$) starting from the maximum weight, the state which is annihilated by the rising operator. Therefore, we need to solve

$$\left(J_{+\ k}^{\ i}\ \delta^j_e + \delta^i_k\ J_{+\ e}^{\ j} \right) \mathcal{E}^{ke}(\bar{P},\lambda) = 0$$

We notice that if we knew the solution to the equation

$$J_{+\ k}^{\ i}\ \mathcal{G}^k = 0$$

then the problem would be solved by $\mathcal{E}^{ke} = \mathcal{G}^k \mathcal{G}^e$. Thus, let us identify $\mathcal{G}^k$.
We recall the SO(3) generator in the adjoint representation
$\left((J^i_{adj})^j{}_k = -i\,\varepsilon^{ijk} \right)$

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad J^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and thus

$$J_+ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ 1 & i & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}$$

Therefore, the solution to $J_+ \mathcal{G} = 0$ is given by

$$\mathcal{G} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}$$

which is an eigenvector of J_3

$$J_3\, \mathcal{G} = +\, \mathcal{G}$$

We may write $\mathcal{G}(\bar{P},+1) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}$. Let us consider the action of the lowering operator on this vector

$$J_-\, \mathcal{G}(\bar{P},+1) = \sqrt{2}\, \mathcal{G}(\bar{P},0) = \sqrt{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and finally

$$J_-\, \mathcal{G}(\bar{P},0) = \sqrt{2}\, \mathcal{G}(\bar{P},-1) = \sqrt{2}\, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

We are now ready to derive the polarization tensors in the particle rest frame (we recall $J_- \mathcal{E}(\bar{P},\lambda) = \sqrt{6-\lambda(\lambda-1)}\ \mathcal{E}(\bar{P},\lambda-1)$)

$$\mathcal{E}(\bar{P},+2) = \mathcal{G}(\bar{P},+1)\, \mathcal{G}(\bar{P}+1)$$

(One can explicitely check that $(J_3 \otimes 1\!\!1 + 1\!\!1 \otimes J_3)\, \mathcal{E}(\bar{P},+2) = +2\, \mathcal{E}(\bar{P},+2)$. In addition, this tensor is traceless.)

$$J_- \mathcal{E}(\overline{P},+2) = 2\, \mathcal{E}(\overline{P},+1) = (J_- \otimes \mathbb{1} + \mathbb{1} \otimes J_-)\, \mathcal{G}(\overline{P},1)\mathcal{G}(\overline{P},1) =$$

$$= \sqrt{2}\, \left(\mathcal{G}(\overline{P},0)\mathcal{G}(\overline{P},1) + \mathcal{G}(\overline{P},1)\mathcal{G}(\overline{P},0) \right)$$

$$\mathcal{E}(\overline{P},+1) = \frac{1}{\sqrt{2}} \left(\mathcal{G}(\overline{P},0)\mathcal{G}(\overline{P},1) + \mathcal{G}(\overline{P},1)\mathcal{G}(\overline{P},0) \right)$$

$$J_- \mathcal{E}(\overline{P},+1) = \sqrt{6}\, \mathcal{E}(\overline{P},0) = \mathcal{G}(\overline{P},-1)\mathcal{G}(\overline{P},1) + 2\,\mathcal{G}(\overline{P},0)\mathcal{G}(\overline{P},0) +$$

$$+ \mathcal{G}(\overline{P},1)\mathcal{G}(\overline{P},-1)$$

$$\mathcal{E}(\overline{P},0) = \frac{1}{\sqrt{6}} \left(\mathcal{G}(\overline{P},-1)\mathcal{G}(\overline{P},1) + 2\,\mathcal{G}(\overline{P},0)\mathcal{G}(\overline{P},0) + \mathcal{G}(\overline{P},1)\mathcal{G}(\overline{P},-1) \right)$$

$$J_- \mathcal{E}(\overline{P},0) = \sqrt{6}\, \mathcal{E}(\overline{P},-1) = \sqrt{3} \left(\mathcal{G}(\overline{P},-1)\mathcal{G}(\overline{P},0) + \mathcal{G}(\overline{P},0)\mathcal{G}(\overline{P},-1) \right)$$

$$\mathcal{E}(\overline{P},-1) = \frac{1}{\sqrt{2}} \left(\mathcal{G}(\overline{P},-1)\mathcal{G}(\overline{P},0) + \mathcal{G}(\overline{P},0)\mathcal{G}(\overline{P},-1) \right)$$

$$J_- \mathcal{E}(\overline{P},-1) = 2\, \mathcal{E}(\overline{P},-2) = 2\, \mathcal{G}(\overline{P},-1)\mathcal{G}(\overline{P},-1)$$

$$\mathcal{E}(\overline{P},-2) = \mathcal{G}(\overline{P},-1)\mathcal{G}(\overline{P},-1)$$

Summarizing, we have

$$\mathcal{E}(\overline{P},+2) = \mathcal{G}(\overline{P},1)\mathcal{G}(\overline{P},1) \qquad \mathcal{E}^{\mu\nu}(\overline{P},+2) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E}(\overline{P},+1) = \frac{1}{\sqrt{2}} \left(\mathcal{E}(\overline{P},0)\mathcal{E}(\overline{P},1) + \mathcal{E}(\overline{P},1)\mathcal{E}(\overline{P},0) \right)$$

$$\mathcal{E}^{\mu\nu}(\overline{P},+1) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & -1 & -i & 0 \end{pmatrix}$$

$$\mathcal{E}(\overline{P},0) = \frac{1}{\sqrt{6}} \left(\mathcal{G}(\overline{P},-1)\mathcal{G}(\overline{P},+1) + 2\,\mathcal{G}(\overline{P},0)\mathcal{G}(\overline{P},0) + \mathcal{G}(\overline{P},1)\mathcal{G}(\overline{P},-1) \right)$$

$$\mathcal{E}^{\mu\nu}(\overline{P},0) = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\mathcal{E}(\overline{P},-1) = \frac{1}{\sqrt{2}} \left(\mathcal{G}(\overline{P},-1)\mathcal{G}(\overline{P},0) + \mathcal{G}(\overline{P},0)\mathcal{G}(\overline{P},-1) \right)$$

$$\mathcal{E}^{\mu\nu}(\bar{P},-1) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \end{pmatrix}$$

$$\mathcal{E}(\bar{P},-2) = \epsilon(\bar{P},-1)\,\epsilon(\bar{P},-1)$$

$$\mathcal{E}^{\mu\nu}(\bar{P},-2) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We notice that these tensors satisfy the following normalization condition

$$\mathcal{E}^{\mu\nu *}(\bar{P},\lambda)\,\mathcal{E}_{\mu\nu}(\bar{P},\lambda') = \delta_{\lambda\lambda'}$$

In addition, as the method of the maximum weight gurantees, all these tensors are traceless. Let us now come back to their constraints. We have seen that

$$\mathcal{E}^{0\mu}(\bar{P},\lambda) = 0$$

$$\mathcal{E}^{\mu}{}_{\mu}(\bar{P},\lambda) = 0$$

We can rewrite these conditions in the following form

$$\bar{P}_\mu\,\mathcal{E}^{\mu\nu}(\bar{P},\lambda) = 0$$

$$\mathcal{E}^{\mu}{}_{\mu}(\bar{P},\lambda) = 0$$

which are suitable to be made covariant.

2. We are interested in writing the constraints of the previous point in a generic reference frame. For this purpose, let us recall that

$$\mathcal{E}^{\mu\nu}(P,\lambda) = \langle 0|\, h^{\mu\nu}(0)\,|P,\lambda\rangle = \langle 0|\, h^{\mu\nu}\, U(\Lambda_P)\,|\bar{P},\lambda\rangle =$$

$$\langle 0|\, U^\dagger(\Lambda_P)\, h^{\mu\nu}(0)\, U(\Lambda_P)\,|\bar{P},\lambda\rangle = {\Lambda_P}^{\mu}{}_{\rho}\,{\Lambda_P}^{\nu}{}_{\sigma}\,\langle 0|\, h^{\rho\sigma}(0)\,|\bar{P},\lambda\rangle =$$

$$= {\Lambda_P}^{\mu}{}_{\rho}\,{\Lambda_P}^{\nu}{}_{\sigma}\,\mathcal{E}^{\rho\sigma}(\bar{P},\lambda)$$

Therefore, we can rewrite the equations above as

$$(\Lambda_P\bar{P})_\mu\;{\Lambda_P}^{\mu}{}_{\rho}\,{\Lambda_P}^{\nu}{}_{\sigma}\,\mathcal{E}^{\rho\sigma}(\bar{P},\lambda) = 0$$

$$\mathcal{E}^{\rho\sigma}(\bar{P},\lambda)\,{\Lambda_P}^{\mu}{}_{\rho}\,{\Lambda_P}^{\nu}{}_{\sigma}\,\eta_{\mu\nu} = 0$$

that means

$$p_\mu \mathcal{E}^{\mu}(p,\lambda) = 0$$

$$\mathcal{E}^{\mu}{}_{\mu}(p,\lambda) = 0$$

They can be written in terms of the wave functions

$$\psi^{\mu\nu}(x;p,\lambda) = \mathcal{E}^{\mu\nu}(p,\lambda)e^{-ip\cdot x}$$

as

$$\partial_\mu \psi^{\mu\nu}(x;p,\lambda) = 0$$

$$\psi^{\mu}{}_{\mu}(x;p,\lambda) = 0$$

3. Let us now consider the Fierz-Pauli equation

$$\Box\psi_{\mu\nu} - \partial_\sigma\partial_\mu\psi^{\sigma}{}_{\nu} - \partial_\sigma\partial_\nu\psi^{\sigma}{}_{\mu} + \partial_\mu\partial_\nu\psi + \eta_{\mu\nu}(\partial_\lambda\partial_\sigma\psi^{\lambda\sigma} - \Box\psi) + m^2(\psi_{\mu\nu} - \eta_{\mu\nu}\psi) = 0$$

We want to prove that its validity implies the constraints we derived before

$$\partial_\mu \psi^{\mu\nu}(x;p,\lambda) = 0$$

$$\psi^{\mu}{}_{\mu}(x;p,\lambda) = 0$$

Let us contract the equation with $\eta^{\mu\nu}$

$$\Box\psi - \partial_\mu\partial_\nu\psi^{\mu\nu} - \partial_\mu\partial_\nu\psi^{\mu\nu} + \Box\psi + 4(\partial_\mu\partial_\nu\psi^{\mu\nu} - \Box\psi) + m^2(\psi - 4\psi) = 0$$

$$(1+1-4)\Box\psi - (1+1-4)\partial_\mu\partial_\nu\psi - 3m^2\psi = 0$$

Therefore we have

$$\Box\psi - \partial_\mu\partial_\nu\psi^{\mu\nu} + \frac{3}{2}m^2\psi = 0$$

Now, let us contract the FP equation with ∂_μ

$$\Box\partial_\mu\psi^{\mu\nu} - \Box\partial_\mu\psi^{\mu\nu} - \partial^\nu\partial_\lambda\partial_\sigma\psi^{\sigma\lambda} + \Box\partial^\nu\psi + \partial^\nu\partial_\lambda\partial_\sigma\psi^{\lambda\sigma} - \Box\partial^\nu\psi + m^2(\partial_\mu\psi^{\mu\nu} - \partial^\nu\psi) = 0$$

We find

$$\partial_\mu\psi^{\mu\nu} = \partial^\nu\psi$$

Let us consider the two equations together

$$\Box\psi - \partial_\mu\partial_\nu\psi^{\mu\nu} + \frac{3}{2}m^2\psi = 0$$

$$\partial_\mu\psi^{\mu\nu} = \partial^\nu\psi$$

Substituting the second in the first one we find

$$\psi = 0$$

$$\partial_\mu\psi^{\mu\nu} = 0$$

which are exactly the constraints we derived before.

4. Let us recall, from the previous exercise, the operators

$$\Pi^L_{\mu\nu} = \frac{\partial_\mu\partial_\nu}{\Box} \qquad \Pi^T_{\mu\nu} = \eta_{\mu\nu} - \Pi^L_{\mu\nu}$$

We want to construct the projectors $\Pi^{(2)}$ $\Pi^{(1)}$ $\Pi^{(0_1)}$ $\Pi^{(0_2)}$ to the spin 2, 1 and 0 (two of them) representation of SO(3).
For this purpose, let us show how to extract the SO(3) irreducible components, that we identified at the beginning of this exercise, from the $(1,1)\oplus(0,0)$ tensor.
We can start working in the particle rest frame and then we will use Lorentz covariance to generalize the results to an arbitrary frame.

- SPIN 0_1

This is the $\mathcal{E}^{00}$ component which can be extracted using the vector $\bar{P}^\mu = (m, \vec{0})$ as

$$\tilde{\Pi}^{(0_1)}_{\mu\nu,\lambda\rho}(\bar{P})\,\mathcal{E}^{\lambda\rho}(\bar{P},\sigma) = \frac{\bar{P}_\mu\bar{P}_\nu}{m^2}\,\frac{\bar{P}_\lambda\bar{P}_\rho}{m^2}\,\mathcal{E}^{\lambda\rho}(\bar{P},\sigma)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- SPIN 0_2

This is the trace $\mathcal{E}^i{}_i(\bar{P},\sigma)$ which can be derived removing the $\mathcal{E}^{00}$ component from $\mathcal{E}^\mu{}_\mu(\bar{P},\sigma)$. Thus, we find

$$\tilde{\Pi}^{(0_2)}_{\mu\nu,\lambda\rho}(\bar{P})\,\mathcal{E}^{\lambda\rho}(\bar{P},\sigma) = \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\bar{P}_\mu\bar{P}_\nu}{m^2}\right)\left(\eta_{\lambda\rho} - \frac{\bar{P}_\lambda\bar{P}_\rho}{m^2}\right)\mathcal{E}^{\lambda\rho}(\bar{P},\sigma)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It takes the (spatial) trace $\mathcal{E}^i{}_i(\bar{P},\sigma)$

- SPIN 1

This is the vector $\mathcal{E}^{0i}(\overline{P},\sigma) = \mathcal{E}^{i0}(\overline{P},\sigma)$. We can extrat these components frau $\mathcal{E}^{\mu\nu}(\overline{P},\sigma)$ as

$$\widetilde{\Pi}^{(1)}_{\mu\nu,\lambda\rho}(\overline{P})\,\mathcal{E}^{\lambda\rho}(\overline{P},\sigma) =$$

$$\left(\frac{\overline{P}_\mu \overline{P}_\lambda}{m^2}\eta_{\nu\rho} + \eta_{\mu\lambda}\frac{\overline{P}_\nu \overline{P}_\rho}{m^2} - 2\,\frac{\overline{P}_\mu \overline{P}_\lambda}{m^2}\,\frac{\overline{P}_\nu \overline{P}_\rho}{m^2}\right)\mathcal{E}^{\lambda\rho}(\overline{P},\sigma) =$$

$$\left[\frac{\overline{P}_\mu \overline{P}_\lambda}{m^2}\left(\eta_{\nu\rho} - \frac{\overline{P}_\nu \overline{P}_\rho}{m^2}\right) + \left(\eta_{\mu\lambda} - \frac{\overline{P}_\mu \overline{P}_\lambda}{m^2}\right)\frac{\overline{P}_\nu \overline{P}_\rho}{m^2}\right]\mathcal{E}^{\lambda\rho}(\overline{P},\sigma)$$

- SPIN 2

Finally, we can derive the spin 2 tensor by subtracting the previous components to $\mathcal{E}^{\mu\nu}(\overline{P},\lambda)$. We find

$$\widetilde{\Pi}^{(2)}_{\mu\nu,\lambda\rho}(\overline{P})\,\mathcal{E}^{\lambda\rho}(\overline{P},\sigma) =$$

$$\Bigg[\Bigg(\eta_{\mu\lambda}\eta_{\nu} - \frac{P_\mu \overline{P}_\nu}{m^2}\,\frac{\overline{P}_\lambda \overline{P}_\rho}{m^2} +$$

$$-\left(\frac{\overline{P}_\mu \overline{P}_\lambda}{m^2}\eta_{\nu\rho} + \eta_{\mu\lambda}\frac{\overline{P}_\nu \overline{P}_\rho}{m^2} - 2\,\frac{\overline{P}_\mu \overline{P}_\lambda}{m^2}\,\frac{\overline{P}_\nu \overline{P}_\rho}{m^2}\right)$$

$$-\frac{1}{3}\left(\eta_{\mu\nu} - \frac{\overline{P}_\mu \overline{P}_\nu}{m^2}\right)\left(\eta_{\lambda\rho} - \frac{\overline{P}_\lambda \overline{P}_\rho}{m^2}\right)\Bigg]\mathcal{E}^{\lambda\rho}(\overline{P},\sigma) =$$

$$\Bigg[\left(\eta_{\mu\lambda}\eta_{\nu\rho} - \frac{\overline{P}_\mu \overline{P}_\lambda}{m^2}\eta_{\nu\rho} - \eta_{\mu\lambda}\frac{\overline{P}_\nu \overline{P}_\rho}{m^2} + \frac{\overline{P}_\mu \overline{P}_\lambda}{m^2}\,\frac{\overline{P}_\nu \overline{P}_\rho}{m^2}\right)$$

$$-\frac{1}{3}\left(\eta_{\mu\nu} - \frac{\overline{P}_\mu \overline{P}_\nu}{m^2}\right)\left(\eta_{\lambda\rho} - \frac{\overline{P}_\lambda \overline{P}_\rho}{m^2}\right)\Bigg]\mathcal{E}^{\lambda\rho}(\overline{P},\sigma) =$$

$$= \left[\left(\eta_{\mu\lambda} - \frac{\overline{P}_\mu \overline{P}_\lambda}{m^2}\right)\left(\eta_{\nu\rho} - \frac{\overline{P}_\nu \overline{P}_\rho}{m^2}\right) - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\overline{P}_\mu \overline{P}_\nu}{m^2}\right)\left(\eta_{\lambda\rho} - \frac{\overline{P}_\lambda \overline{P}_\rho}{m^2}\right)\right]\mathcal{E}^{\lambda\rho}(\overline{P},\sigma)$$

Now we recall that

$$\widetilde{\Pi}^{L}_{\mu\nu}(\overline{P}) = \frac{\overline{P}_\mu \overline{P}_\nu}{m^2} \qquad \widetilde{\Pi}^{T}_{\mu\nu}(\overline{P}) = \eta_{\mu\nu} - \frac{\overline{P}_\mu \overline{P}_\nu}{m^2}$$

where

$$\widetilde{\Pi}^{L}_{\mu\nu}(P) = e^{+iP\cdot x}\,\Pi^{L}_{\mu\nu}(\partial)\,e^{-iP\cdot x}$$

therefore, we find the relations

$$\tilde{\Pi}^{(0_1)}_{\mu\nu,\lambda\rho}(\bar{P}) = \tilde{\Pi}^{L}_{\mu\nu}(\bar{P})\,\tilde{\Pi}^{L}_{\lambda\rho}(\bar{P})$$

$$\tilde{\Pi}^{(0_2)}_{\mu\nu,\lambda\rho}(\bar{P}) = \frac{1}{3}\,\tilde{\Pi}^{T}_{\mu\nu}(\bar{P})\,\tilde{\Pi}^{T}_{\lambda\rho}(\bar{P})$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\lambda\rho}(\bar{P}) = \tilde{\Pi}^{L}_{\mu\lambda}(\bar{P})\,\tilde{\Pi}^{T}_{\nu\rho}(\bar{P}) + \tilde{\Pi}^{T}_{\mu\lambda}(\bar{P})\,\tilde{\Pi}^{L}_{\nu\rho}(\bar{P})$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\lambda\rho}(\bar{P}) = \tilde{\Pi}^{T}_{\mu\lambda}(\bar{P})\,\tilde{\Pi}^{T}_{\nu\rho}(\bar{P}) - \frac{1}{3}\,\tilde{\Pi}^{T}_{\mu\nu}(\bar{P})\,\tilde{\Pi}^{T}_{\lambda\rho}(\bar{P})$$

These equations are relations between tensors and thus they do not depend on the particular coordinate system (we can transform both side of the equation in the same way to rewrite the relation in a different frame)

Thus, we have

$$\tilde{\Pi}^{(0_1)}_{\mu\nu,\lambda\rho}(P) = \tilde{\Pi}^{L}_{\mu\nu}(P)\,\tilde{\Pi}^{L}_{\lambda\rho}(P)$$

$$\tilde{\Pi}^{(0_2)}_{\mu\nu,\lambda\rho}(P) = \frac{1}{3}\,\tilde{\Pi}^{T}_{\mu\nu}(\bar{P})\,\tilde{\Pi}^{T}_{\lambda\rho}(P)$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\lambda\rho}(P) = \tilde{\Pi}^{L}_{\mu\lambda}(P)\,\tilde{\Pi}^{T}_{\nu\rho}(P) + \tilde{\Pi}^{T}_{\mu\lambda}(P)\,\tilde{\Pi}^{L}_{\nu\rho}(P)$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\lambda\rho}(P) = \tilde{\Pi}^{T}_{\mu\lambda}(P)\,\tilde{\Pi}^{T}_{\nu\rho}(P) - \frac{1}{3}\,\tilde{\Pi}^{T}_{\mu\nu}(P)\,\tilde{\Pi}^{T}_{\lambda\rho}(P)$$

which is the desired result in momentum space.

We notice that $\Pi^{(\alpha)}_{\mu\nu,\lambda\rho}$ are projectors as a consequence of the fact that $\Pi^{L/T}_{\mu\nu}$ are projectors. In particular, they satisfy the completeness relation

$$\Pi^{(0_1)}_{\mu\nu,\lambda\rho} + \Pi^{(0_2)}_{\mu\nu,\lambda\rho} + \Pi^{(1)}_{\mu\nu,\lambda\rho} + \Pi^{(2)}_{\mu\nu,\lambda\rho} = \eta_{\mu\nu}\,\eta_{\lambda\rho}$$

Finally, we can define the projectors that mix the two spin 0 representations

$$\Pi^{(0_{12})}_{\mu\nu,\lambda\sigma} = \frac{1}{\sqrt{3}}\,\Pi^{L}_{\mu\nu}\,\Pi^{T}_{\lambda\sigma}$$

$$\Pi^{(0_{21})}_{\mu\nu,\lambda\sigma} = \frac{1}{\sqrt{3}}\,\Pi^{T}_{\mu\nu}\,\Pi^{L}_{\lambda\sigma}$$

Their action is clear when we Fourier transform the operators and we consider the particle rest frame.

$$\tilde{\Pi}^{(0_{12})}_{\mu\nu,\lambda\sigma}(\bar{P}) = \frac{1}{\sqrt{3}}\,\underbrace{\frac{\bar{P}_\mu\bar{P}_\nu}{m^2}}_{\text{It places the trace in the (0,0) component of the tensor}}\,\underbrace{\left(\eta_{\lambda\sigma} - \frac{\bar{P}_\lambda\bar{P}_\sigma}{m^2}\right)}_{\text{It takes the (spatial) trace}}$$

$$\tilde{\Pi}^{(0_{21})}_{\mu\nu,\lambda\sigma} = \frac{1}{\sqrt{3}}\,\left(\eta_{\mu\nu} - \frac{\bar{P}_\mu\bar{P}_\nu}{m^2}\right)\,\overset{\text{It takes the (0,0) component of the tensor}}{\frac{\bar{P}_\lambda\bar{P}_\sigma}{m^2}}$$

It places this component in the (spatial) diagonal.

We notice that these operators satisfy

$$\Pi^{(0_{12})}_{\mu\nu,\lambda\sigma}\,\Pi^{(0_{21})\,\lambda\sigma}{}_{,\rho\tau} = \frac{1}{3}\,\Pi^{L}_{\mu\nu}\Pi^{T}_{\lambda\sigma}\,\Pi^{T\,\lambda\sigma}\Pi^{L}_{\rho\tau} = \Pi^{L}_{\mu\nu}\Pi^{L}_{\rho\tau} = \Pi^{(0_1)}_{\mu\nu,\rho\tau}$$

$$\Pi^{(0_{21})}_{\mu\nu,\lambda\sigma}\,\Pi^{(0_{12})\,\lambda\sigma}{}_{,\rho\tau} = \frac{1}{3}\,\Pi^{T}_{\mu\nu}\Pi^{L}_{\lambda\sigma}\,\Pi^{L\,\lambda\sigma}\Pi^{T}_{\rho\tau} = \frac{1}{3}\,\Pi^{T}_{\mu\nu}\Pi^{T}_{\rho\tau} = \Pi^{(0_2)}_{\mu\nu,\rho\tau}$$

5. We want to rewrite the Fierz-Pauli equation in terms of the projectors we found before. Let us start writing

$$\big(\Box\,\eta_{\lambda\mu}\eta_{\rho\nu} - \eta_{\rho\nu}\partial_\lambda\partial_\mu - \eta_{\lambda\mu}\partial_\rho\partial_\nu + \eta_{\lambda\rho}\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial_\lambda\partial_\rho - \Box\,\eta_{\mu\nu}\eta_{\rho\lambda}$$
$$+\, m^2(\eta_{\lambda\mu}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\lambda})\big)\,\psi^{\rho\lambda} = 0$$

or equivalently

$$\left[(\Box + m^2)(\eta_{\lambda\mu}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\lambda}) - \eta_{\rho\nu}\partial_\lambda\partial_\mu - \eta_{\lambda\mu}\partial_\rho\partial_\nu + \eta_{\lambda\rho}\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial_\lambda\partial_\rho\right]\psi^{\rho\lambda} = 0$$

We now use the following facts

$$\Pi^{L}_{\mu\nu} + \Pi^{T}_{\mu\nu} = \eta_{\mu\nu}$$

$$\Box\,\Pi^{L}_{\mu\nu} = \partial_\mu\partial_\nu$$

$$\Pi^{(0_1)}_{\mu\nu,\lambda\rho} + \Pi^{(0_2)}_{\mu\nu,\lambda\rho} + \Pi^{(1)}_{\mu\nu,\lambda\rho} + \Pi^{(2)}_{\mu\nu,\lambda\rho} = \eta_{\mu\lambda}\eta_{\nu\rho}$$

Let us consider the first term

$$(\Box + m^2)(\eta_{\lambda\mu}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\lambda}) =$$

$$= (\Box + m^2)\big((\Pi^{L}_{\mu\lambda} + \Pi^{T}_{\mu\lambda})(\Pi^{L}_{\rho\nu} + \Pi^{T}_{\rho\nu}) - (\Pi^{L}_{\mu\nu} + \Pi^{T}_{\mu\nu})(\Pi^{L}_{\rho\lambda} + \Pi^{T}_{\rho\lambda})\big) =$$

$$= (\Box + m^2)\big(\Pi^{L}_{\mu\lambda}\Pi^{L}_{\rho\nu} + \Pi^{L}_{\mu\lambda}\Pi^{T}_{\rho\nu} + \Pi^{T}_{\mu\lambda}\Pi^{L}_{\rho\nu} + \Pi^{T}_{\mu\lambda}\Pi^{T}_{\rho\nu} +$$
$$- \Pi^{L}_{\mu\nu}\Pi^{L}_{\rho\lambda} - \Pi^{L}_{\mu\nu}\Pi^{T}_{\rho\lambda} - \Pi^{T}_{\mu\nu}\Pi^{L}_{\rho\lambda} - \Pi^{T}_{\mu\nu}\Pi^{T}_{\rho\lambda}\big) =$$

$$\Pi^{L}_{\mu\lambda}\Pi^{L}_{\rho\nu} = \Pi^{L}_{\mu\nu}\Pi^{L}_{\rho\lambda}$$

$$= (\Box + m^2)\big(\Pi^{L}_{\mu\lambda}\Pi^{T}_{\rho\nu} + \Pi^{T}_{\mu\lambda}\Pi^{L}_{\rho\nu} - \Pi^{L}_{\mu\nu}\Pi^{T}_{\rho\lambda} - \Pi^{T}_{\mu\nu}\Pi^{L}_{\rho\lambda}$$
$$+ \Pi^{T}_{\mu\lambda}\Pi^{T}_{\rho\nu} - \Pi^{T}_{\mu\nu}\Pi^{T}_{\rho\lambda}\big) =$$

$$= (\Box + m^2)\big(\Pi^{(2)}_{\mu\nu,\lambda\rho} + \Pi^{(1)}_{\mu\nu,\lambda\rho} - \sqrt{3}\,\Pi^{(0_{12})}_{\mu\nu,\lambda\rho} - \sqrt{3}\,\Pi^{(0_{21})}_{\mu\nu,\lambda\rho} - 2\,\Pi^{(0_2)}_{\mu\nu,\lambda\rho}\big)$$

Now, we consider the second one

$$-\eta_{\rho\nu}\partial_\lambda\partial_\mu - \eta_{\lambda\mu}\partial_\rho\partial_\nu + \eta_{\lambda\rho}\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial_\lambda\partial_\rho =$$

$$= -\Box(\Pi^L_{\rho\nu} + \Pi^T_{\rho\nu})\Pi^L_{\lambda\mu} - \Box(\Pi^L_{\lambda\mu} + \Pi^T_{\lambda\mu})\Pi^L_{\rho\nu} + \Box(\Pi^T_{\lambda\rho} + \Pi^L_{\lambda\rho})\Pi^L_{\mu\nu} +$$

$$+ \Box(\Pi^T_{\mu\nu} + \Pi^L_{\mu\nu})\Pi^L_{\lambda\rho} =$$

$$= +\Box(-\Pi^L_{\rho\nu}\Pi^L_{\lambda\mu} - \Pi^T_{\rho\nu}\Pi^L_{\lambda\mu} - \Pi^L_{\lambda\mu}\Pi^L_{\rho\nu} - \Pi^T_{\lambda\mu}\Pi^L_{\rho\nu} + \Pi^T_{\lambda\rho}\Pi^L_{\mu\nu}$$

$$+ \Pi^L_{\lambda\rho}\Pi^L_{\mu\nu} + \Pi^T_{\mu\nu}\Pi^L_{\lambda\rho} + \Pi^L_{\mu\nu}\Pi^L_{\lambda\rho}) =$$

The $\Pi^L\Pi^L$ terms cancel

$$= \Box(-\Pi^T_{\rho\nu}\Pi^L_{\lambda\mu} - \Pi^T_{\lambda\mu}\Pi^L_{\rho\nu} + \Pi^T_{\lambda\rho}\Pi^L_{\mu\nu} + \Pi^T_{\mu\nu}\Pi^L_{\lambda\rho}) =$$

$$= \Box(\sqrt{3}\,\Pi^{(0_{21})}_{\mu\nu,\lambda\rho} + \sqrt{3}\,\Pi^{(0_{12})}_{\mu\nu,\lambda\rho} - \Pi^{(1)}_{\mu\nu,\lambda\rho})$$

Therefore we found that the Fierz-Pauli equation can be written as

$$[(\Box+m^2)(\Pi^{(2)}_{\mu\nu,\lambda\rho} - 2\Pi^{(0_2)}_{\mu\nu,\lambda\rho}) + m^2(\Pi^{(1)}_{\mu\nu,\lambda\rho} - \sqrt{3}\,\Pi^{(0_{12})}_{\mu\nu,\lambda\rho} - \sqrt{3}\,\Pi^{(0_{21})}_{\mu\nu,\lambda\rho})]\psi^{\lambda\rho} = 0$$

We can verify the result replacing the projectors with their explicit expression.

$$(\Box+m^2)(\Pi^{(2)}_{\mu\nu,\lambda\rho} - 2\Pi^{(0_2)}_{\mu\nu,\lambda\rho}) =$$

$$= (\Box+m^2)(\Pi^T_{\mu\lambda}\Pi^T_{\rho\nu} - \frac{1}{3}\Pi^T_{\mu\nu}\Pi^T_{\lambda\rho} - \frac{2}{3}\Pi^T_{\mu\nu}\Pi^T_{\lambda\rho}) =$$

$$= (\Box+m^2)\left(\left(\eta_{\mu\lambda} - \frac{\partial_\mu\partial_\lambda}{\Box}\right)\left(\eta_{\rho\nu} - \frac{\partial_\rho\partial_\nu}{\Box}\right) - \left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\left(\eta_{\rho\lambda} - \frac{\partial_\rho\partial_\lambda}{\Box}\right)\right) =$$

$$= (\Box+m^2)\left(\eta_{\mu\lambda}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\lambda}\frac{\partial_\rho\partial_\nu}{\Box} - \eta_{\rho\nu}\frac{\partial_\mu\partial_\lambda}{\Box} + \frac{\partial_\mu\partial_\lambda\partial_\rho\partial_\nu}{\Box^2}\right.$$

$$\left.+ \eta_{\mu\nu}\frac{\partial_\rho\partial_\lambda}{\Box} + \eta_{\rho\lambda}\frac{\partial_\mu\partial_\nu}{\Box} - \frac{\partial_\rho\partial_\lambda\partial_\mu\partial_\nu}{\Box^2}\right) =$$

$$= (\Box+m^2)(\eta_{\mu\lambda}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\lambda}) - \eta_{\mu\lambda}\partial_\rho\partial_\nu - \eta_{\rho\nu}\partial_\mu\partial_\lambda + \eta_{\mu\nu}\partial_\rho\partial_\lambda + \eta_{\rho\lambda}\partial_\mu\partial_\nu$$

$$+ m^2\left(-\eta_{\mu\lambda}\frac{\partial_\rho\partial_\nu}{\Box} - \eta_{\rho\nu}\frac{\partial_\mu\partial_\lambda}{\Box} + \eta_{\mu\nu}\frac{\partial_\rho\partial_\lambda}{\Box} + \eta_{\rho\lambda}\frac{\partial_\mu\partial_\nu}{\Box}\right)$$

$$m^2 \left(\Pi^{(1)}_{\mu\nu,\lambda\rho} - \sqrt{3}\, \Pi^{(0_{12})}_{\mu\nu,\lambda\rho} - \sqrt{3}\, \Pi^{(0_{21})}_{\mu\nu,\lambda\rho} \right) \Big] =$$

$$= m^2 \left(\Pi^{T}_{\mu\lambda} \Pi^{L}_{\nu\rho} + \Pi^{L}_{\mu\lambda} \Pi^{T}_{\nu\rho} - \Pi^{L}_{\mu\nu} \Pi^{T}_{\lambda\rho} - \Pi^{T}_{\mu\nu} \Pi^{L}_{\lambda\rho} \right) =$$

$$= m^2 \Big(\eta_{\mu\lambda} \frac{\partial_\nu \partial_\rho}{\Box} - \frac{\partial_\mu \partial_\lambda \partial_\nu \partial_\rho}{\Box^2} + \eta_{\nu\rho} \frac{\partial_\mu \partial_\lambda}{\Box} - \frac{\partial_\mu \partial_\lambda \partial_\nu \partial_\rho}{\Box^2} +$$

$$- \frac{\partial_\mu \partial_\nu}{\Box} \eta_{\lambda\rho} + \frac{\partial_\mu \partial_\nu \partial_\lambda \partial_\rho}{\Box^2} - \eta_{\mu\nu} \frac{\partial_\lambda \partial_\rho}{\Box} + \frac{\partial_\mu \partial_\nu \partial_\lambda \partial_\rho}{\Box^2} \Big) =$$

$$= m^2 \left(\eta_{\mu\lambda} \frac{\partial_\nu \partial_\rho}{\Box} + \eta_{\nu\rho} \frac{\partial_\mu \partial_\lambda}{\Box} - \eta_{\lambda\rho} \frac{\partial_\mu \partial_\nu}{\Box} - \eta_{\mu\nu} \frac{\partial_\lambda \partial_\rho}{\Box} \right)$$

The non local terms cancel and we are left with the FP equation.